Species cohabitation

draft written by Bernard Beauzamy, May 14, 2005.

Let us assume that three species live in the same place, for example a lake.
Let's denote these species by A, B, C.

We are interested in the interactions, which may be of various nature (one may eat another, or need another, ...). We define and fix specific interactions, and we will try to build models which indicate the growth of the population, taking into account these interactions.

First approach : in terms of differential systems of equations.

We write that the "rate of increase" of the species A depends linearly on the population of the three species. This gives us an equation of the form:

$$\frac{dA}{dt} = a_1 A + b_1 B + c_1 C$$

for some coefficients a_1, b_1, c_1 which are real numbers. Some may be positive (meaning that it helps the reproduction) ; some may be negative (it contradicts the reproduction).

And similarly for the two others :

$$\frac{dB}{dt} = a_2 A + b_2 B + c_2 C$$

$$\frac{dC}{dt} = a_3 A + b_3 B + c_3 C$$

Let X be the column-vector $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ and let U be the 3x3 matrix of the coefficients. We then get a system:

$$\frac{dX}{dt} = UX \quad (1)$$

There is a large literature about such systems (semi-groups of operators). The asymptotic behavior depends on the spectrum of U, which is easy to find in our case.

Make sure to describe completely what are the interactions between species. Make all assumptions explicit.

Give precise value to the coefficients in the matrix X, so as to reflect a specific situation, and give precise, numerical values to the initial situation.

Question 1

Using results from this literature, can we predict all possible asymptotic behaviors, depending on all initial conditions ? Two types only seem possible :

- some equilibrium ;
- one species disappears totally.

Try to investigate the "robustness" of the equilibrium : is it stable or not ? Stable means that, in the case of small deviations, one comes back to it.

Investigate also the rate at which the equilibrium or disappearance occur ; most likely, this is going to be exponential.

Question 2

Using results from Question 1, criticize the model given by (1).

First, it is linear, and in real life nothing is truly linear. Of course, locally, everything can be linearized, but not for long times. Model (1) gives an asymptotic behavior, plus rates of increase/decrease : is this information coherent with what is observed for true species ?

The problem is here with the notion of derivative, which leads to differential equations. The derivative might not be the right tool. It denotes, seemingly, an instantaneous rate of growth, but the overall rigidity of the model turns it into a global rate of growth.

Fluctuations : Nature tends to have fluctuations between species, not equilibrium. Is it possible that model (1) may be used to describe fluctuations ? This is not likely to be so : in a model such as (1), things converge or diverge to infinity : they cannot fluctuate.

From Questions 1 and 2, should come out a clear answer to this question : are linear-differential models such as (1) proper to describe cohabitation between species. The fact they are used by many people does not give any support to their credibility. They were used because of an apparent "simplicity", but in most cases they give results which 1) can be computed only numerically 2) do not reflect reality.

Second approach : the next stage as a function of present.

Let's now change to a radically different approach. Let A_n be the quantity of individuals in species A, and similarly for others.

Let's compute the quantity A_{n+1} by an expression of the form:

$$A_{n+1} = f(A_n, B_n, C_n)$$

and similarly :

$$B_{n+1} = g(A_n, B_n, C_n)$$

$$C_{n+1} = h(A_n, B_n, C_n)$$

which mean that the next state depends only on the present state, not on the past ones.

From the interactions defined originally (one species eats another, and so on), try to figure out what the properties of the functions f, g, h should be (for instance, increasing with respect to some variable, up to some threshold, decreasing after that threshold, and so on).

Do not try to find specific forms for these functions (such as linear, quadratic, and so on), but instead try to answer the following question : what are the most general properties which completely describe the initial interactions we have chosen ?

When this is done, try to answer the question : what are the asymptotic behaviors of such functions ? Do they necessary lead to equilibrium or extinction ? Are fluctuations possible ?